Math 10B with Professor Stankova
Worksheet, Discussion #5; Thursday, 2/1/2018
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Distinguishable Boxes

Examples

1. Suppose I am catering from Yali's and want to buy sandwiches to feed 60 students. How many ways can I do this if they have 8 sandwich options? How many ways can I do this if I want to get at least 2 of each sandwich?

> **Solution:** We can think of putting indistinguishable balls into boxes, one box for each sandwich option. Thus, we need 8 boxes which is 7 dividers. Thus, there are $\binom{67}{7}$ ways to do this.
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> If I want at least 2 of each sandwich, what I can do is buy two of each sandwich first, then I only need to buy 44 more sandwiches and I can do this $\binom{44+7}{7} = \binom{51}{7}$ different ways.

2. How many ways can I distribute 30 course spots amongst 4 grades (freshmen, sophomores, juniors, seniors) so that there are no more than 10 freshmen in the course?

> **Solution:** We already know how to do this if we require there to be more than 10 freshmen in the course because we just put 11 freshmen in the course and then do our regular distributing of indistinguishable balls into distinguishable boxes. Doing this, after putting 11 freshmen in, there are a total of $\binom{19+4-1}{4-1} = \binom{22}{3}$ different ways. Thus, there are $\binom{22}{3}$ ways for this to fail and in order to count the right number of ways, we can use complementary counting. There are $\binom{30+4-1}{4-1} = \binom{33}{3}$ different ways to do this regularly. Therefore, there are $\binom{33}{3} - \binom{22}{3}$ ways to do this in order for there to be at most 10 freshmen in the course.

Problems

3. True **FALSE** When we are counting the number of bitstrings of 0 and 1 with a certain number of 0's, the ordering of the 0 and 1's matter which means that we should use $P()$ as opposed to $C()$.

> **Solution:** We say order doesn't matter to mean that when picking the **location** of the 0's, the order in which we pick them does not matter and hence we should use C. So order always has to do with the order in which you choose things, not in the total relative ordering compared to everything else.

4. **TRUE** False In Example 1, since each student is getting a sandwich, the balls are sandwiches and the urns are students.

> **Solution:** It is the opposite way around because we want to know the distribution of how many students get a particular type of sandwich, so how many balls are in a particular urn.

5. In the card game Sheng Ji or 80 points, two decks of a total of 108 cards are dealt out to 4 people such that each person gets 25 cards and there are 8 cards left over. How many ways can this occur?

> **Solution:** There are $\binom{108}{25}$ ways to choose the cards for the first person. Then there are $\binom{83}{25}$ ways to choose the cards for the second, $\binom{58}{25}$ for the third and $\binom{33}{25}$ for the last. This gives a total amount of
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> $$\binom{108}{25}\binom{83}{25}\binom{58}{25}\binom{33}{25} = \binom{108}{25, 25, 25, 25}.$$

6. How many solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 20$ if all are positive integers and $x_3 \leq 3$?

> **Solution:** First, we are distributing 20 balls into 5 boxes such that the third box as at most 3 balls and all the boxes have at least one ball. We can do this by first distributing one ball into each, so we have 15 left to distribute, and the third can have at most 2 more. We do this via complementary counting. There are a total of $\binom{15+5-1}{5-1} = \binom{19}{4}$ ways and the ways that fail are if we put 3 more balls into the third to have 12 left to distribute in $\binom{12+5-1}{5-1} = \binom{16}{4}$ ways. So there are a total of $\binom{19}{4} - \binom{16}{4}$ ways.

7. How many 7 digit decreasing numbers are there? One example is 9777650.

Solution: This is the number of ways to choose 7 digits out of 10 total with repetition. There are $\binom{7+10-1}{10-1} = \binom{16}{9}$ ways to do this. But, if we choose all 0's, then this is not a 7 digit number so we have to get rid of that case and that is the only bad case. So, there are a total of $\binom{16}{9} - 1$ total ways.

8. How many 3 digit numbers have a sum of digits equal to 9?

Solution: This is asking how many ways can we have $x_1 + x_2 + x_3 = 9$ if $x_1 \geq 1$ because it has to be a 3 digit number. This is the same as $x_1 + x_2 + x_3 = 8$ without any restrictions and this can be done $\binom{8+3-1}{3-1} = \binom{10}{2}$ ways.

9. How many numbers less than $1,000,000$ have the sum of their digits equal to 10?

Solution: This is asking how many ways can we have $x_1 + x_2 + \cdots + x_6 = 10$. There are $\binom{10+6-1}{6-1} = \binom{15}{5}$ ways to do this. But 6 of those ways have $x_i = 10$ which is not possible. Therefore, there are a total of $\binom{15}{5} - 6$ ways.

10. How many ways can you deal the 52 cards of a deck to 4 people so that everyone gets 13 cards and the oldest player gets the queen of spades?

Solution: We can assign the oldest player the queen of spades and then there are $\binom{51}{12}$ ways to assign the rest of his/her cards. For the other people, there are $\binom{39}{13}$, $\binom{26}{13}$ and $\binom{13}{13}$ ways. Thus there are a total of

$$\binom{51}{12}\binom{39}{13}\binom{26}{13}\binom{13}{13} = \binom{51}{12, 13, 13, 13}$$

ways.

Indistinguishable Boxes

Example

11. How many ways are there to split 28 distinct students up into at most 6 different groups if the groups are not numbered? What if the students are not distinct?

Solution: Here, the groups will be the urns and students the balls. There are distinct balls and indistinguishable urns. There can be $1, 2, 3, 4, 5$ or 6 groups. There are going to be $S(28, 1) + S(28, 2) + S(28, 3) + S(28, 4) + S(28, 5) + S(28, 6)$ ways to split the students up into at most 6 groups.

If the students are not distinct, then the balls are indistinguishable and this can be done in $p_1(28) + p_2(28) + p_3(28) + p_4(28) + p_5(28) + p_6(28)$ ways.

Problems

12. True **FALSE** The only way to determine what $S(5, 3)$ is to list out all the possibilities.

Solution: There is a recursive formula $S(n+1, k) = kS(n, k) + S(n, k-1)$ that we can use to figure out what it is.

13. **TRUE** False The only way to determine what $p_3(5)$ is to list out all the possibilities.

Solution: This is the only way you know how to for now.

14. True **FALSE** In order to determine the number of ways to distribute 10 distinguishable items into 3 identical boxes so that each box has at least 2 items, we can place one item in each box and this problem reduces to the regular case of distributing 7 items in 3 identical boxes which is $S(7, 3)$ ways.

Solution: We can only use this strategy if the items are indistinguishable so that it doesn't matter which items we put in the boxes.

15. **TRUE** False In order to determine the number of ways to distribute 10 identical items into 3 identical boxes so that each box has at least 2 items, we can place one item in each box and this problem reduces to the regular case of distributing 7 items in 3 identical boxes which is $p_3(7)$ ways.

16. There are 14 students that want to break off into 3 non-empty study groups. How many ways can this occur?

Solution: $S(14, 3)$.

17. I want to store my 200 Yu-gi-oh cards in 4 different identical boxes. How many ways can I do this if some boxes are allowed to be empty?

Solution: You can do this $S(200, 1) + S(200, 2) + S(200, 3) + S(200, 4)$ ways.

18. How many ways are there to split 15 identical marbles to 5 different non-empty groups?

Solution: There are $p_5(15)$ different ways to do this.

19. How many ways are there of distributing 30 identical objects into 3 boxes if each box must have at least 5 items?

Solution: We can put 4 of each item in each box and then this reduces to the regular case. Thus, we need to distribute 18 objects into 3 boxes, which can be done $p_3(18)$ ways.